ANALYTICAL SOLUTIONS OF NONSTATIONARY ADIABATIC PROCESSES FOR COMPRESSION (EXTENSION) OF VISCOPLASTIC SPHERICAL AND CYLINDRICAL SHELLS, SPHERICAL AND CYLINDRICAL LAYERS FROM VISCOUS LIQUID

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Summery. In first part of this paper present original analytical solutions for one-dimensional problems of adiabatic compression and extension of viscoplastic (Sokolovsky-Perzyna type model) thick-walled spherical and cylindrical shells. Solutions obtained in Lagrange coordinates and at hypothesis of incompressibility of shells material. In second part of paper present original analytical solutions for one-dimensional problems of adiabatic compression and extension of viscous (Navier-Stokes model) incompressible spherical and cylindrical layers. These solutions may be, in particular, used for testing computational programs and estimate of effectiveness of new numerical methods.

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1 INTRODUCTION

It is known a few analytical solutions for dynamical problems of elastoviscoplasticity in view of its particular complication (see [1-5] and references in their publications). In contrast to papers [2, 3], where were investigated problems of compression and extension of spherical pores from incompressible viscoplastic material under permanent loading, in first part of present paper solutions obtained under dynamical external loading. In addition analogous solutions were obtained for cylindrical thick-walled shells.

Classical Zababaxin problem about fill out of bubbles in viscous liquid under action of permanent pressure on infinity [6], in second part of present paper this problem generalized for case of dynamical both external and internal pressure. In addition solution obtained for spherical layer finite thickness. Analogous solutions were obtained for cylindrical layer of viscous liquid.

2 COMPRESSION OF SPHERICAL SHELL

In one-dimensional approximation (all parameters dependent on radial Lagrange coordinate R and time t) consider process of adiabatic compression of spherical shell which internal and external radius change in time on low $r_0 = r(R_0, t)$ and $r_1 = r(R_1, t)$ respectively, where R_0 and R_1 - internal and external radius of shell in initial time $t = 0$ (figure 1).

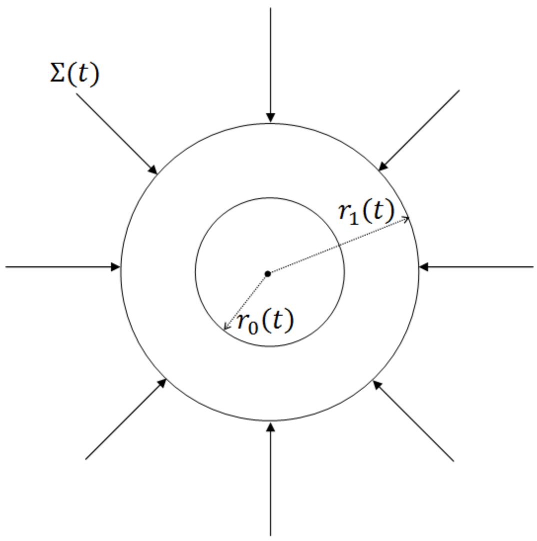

Figure 1

Do next simplifying assimpsion:

1) Behavior of shell material described of equations of elastoviscoplastic model Sokolovsky-Perzyna type [7]:

$$\dot{e}_{ij} = \frac{\dot{S}_{ij}}{2\mu} + \frac{S_{ij}\left(\sqrt{S_{ij}S_{ij}} - \sqrt{\frac{2}{3}}Y_0\right)}{2\eta\sqrt{S_{ij}S_{ij}}} H\left(\sqrt{S_{ij}S_{ij}} - \sqrt{\frac{2}{3}}Y_0\right) \tag{1}$$

Here $\dot{e}_{ij}$ and $S_{ij}-$ deviators of strain rates and stresses; Y_0- yield limit under simple tension; $H(x)-$ Heviside function; μ and $\eta-$shear module and dynamic viscosity.

2) Elastic deformations can be neglected: $\dot{\varepsilon}^e_{ij}=0,\ \dot{\varepsilon}_{ij}=\dot{\varepsilon}^p_{ij}$ $(\dot{\varepsilon}^e_{ij},\dot{\varepsilon}^p_{ij}$, $\dot{\varepsilon}_{ij}=\dot{\varepsilon}^e_{ij}+\dot{\varepsilon}^p_{ij}-$elastic, plastic and total strain rates respectively); plastic flow is incompressible: $\dot{\varepsilon}_{kk}=0.$

Therefore equations (1) reduced to one equation

$$\sigma_R-\sigma_\theta=Y_0+2\eta(\dot{\varepsilon}_R-\dot{\varepsilon}_\theta) \tag{2}$$

Here $v-$radial velocity; $\dot{\varepsilon}_R=\partial v/\partial R,\ \dot{\varepsilon}_\theta=v/R-$radial and ring strain rates; $\sigma_R,\sigma_\theta-$radial and ring stresses.

Hypothesis of material incompressibility $\dot{\varepsilon}_R+2\dot{\varepsilon}_\theta=0$ give equation for determination of velocity distribution in shell:

$$\partial v/\partial R+2v/R=0 \tag{3}$$

Solution of equation (3) has next form:

$$v=C(t)/R^2 \tag{4}$$

where $C(t)\le 0$ because take place compression of shell and velocity $v\le 0$.

The equation of momentum for spherical shell is

$$\rho_0\dot{v}=\frac{\partial\sigma_R}{\partial R}+2\frac{\sigma_R-\sigma_\theta}{R} \tag{5}$$

Here ρ_0-density of shell material; the dot over symbols indicates the material derivative with respect to time.

Put next boundary conditions:

$$\sigma_R\,|_{R=R_1}=\Sigma(t)<0,\qquad \sigma_R\,|_{R=R_0}=0 \tag{6}$$

If magnitude of external loading in initial moment $\Sigma(0)$ more than some critical value, which bill be find below, than material completely passed in plastic state. Precisely this loading is considered.

Substitute (2), (4) to equation of momentum (5), and integrate over R with taking account of boundary conditions (6), we get ordinary differential equation for function $C(t)$:

$$\dot{C}(t)+\alpha C(t)=\beta\Sigma(t)+\gamma \tag{7}$$

where

$$\alpha=\frac{4\eta(R_1^2+R_1R_0+R_0^2)}{\rho_0R_1^2R_0^2},\qquad \beta=\frac{R_1R_0}{\rho_0(R_1-R_0)},\qquad \gamma=\frac{2Y_0R_1R_0}{\rho_0(R_1-R_0)}\ln\frac{R_1}{R_0}$$

Solution of equation (7) with initial condition $C(0)=0$ (velocity of shell under $t=0$ equal zero) is:

$$C(t) = \beta e^{-\alpha t} \int_0^t e^{\alpha t} \Sigma(t) dt + \frac{\gamma}{\alpha}(1 - e^{-\alpha t}) \tag{8}$$

Taking into account that acceleration in initial moment under compression is negative, i.e. $\dot{C}(0) < 0$, we take condition for initial external pressure to shell:

$$\Sigma(0) < -\frac{\gamma}{\beta} = -2Y_0 \ln \frac{R_1}{R_0} \tag{9}$$

Value $\Sigma_{\min} = 2Y_0 \ln \frac{R_1}{R_0}$ is effective yield limit under multifold compression. This result was obtained earlier and in paper [2].

When external pressure to shell in constant, i.e. $\Sigma(t) = \Sigma_0 = const < -2Y_0 \ln \frac{R_1}{R_0}$, we find that

$$C(t) = \frac{1}{\alpha}(\beta \Sigma_0 + \gamma)(1 - e^{-\alpha t}) \tag{10}$$

Current value of Euler coordinate for material particle with Lagrange coordinate R is:

$$r(R,t) = R \left\lfloor 1 + \frac{\beta \Sigma_0 + \gamma}{\alpha R^3} \left(t + \frac{e^{-\alpha t} - 1}{\alpha} \right) \right\rfloor \tag{11}$$

Obtain equation for moment of collapse of incompressible spherical shell $t = t_s^*$ (i.e. then $r(R_0, t_s^*) = 0$), for which obtained solution have physical meaning:

$$e^{-\alpha t_s^*} + \alpha t_s^* = 1 - \frac{R_0^3 \alpha^2}{\beta \Sigma_0 + \gamma} \tag{12}$$

Under condition (9) equation (12) always have uniqueness solution which is not has a form of elemental functions; however numerical solution of equation (12) is not a problem.

3 COMPRESSION OF CYLINDRICAL SHELL

In case of cylindrical shell under made above assumptions and since condition that displacements along axis z are lacking, i.e. strain rates $\dot{\varepsilon}_z = 0$, equations (1) result to one equationility

$$\sigma_R - \sigma_\theta = \frac{2}{\sqrt{3}} Y_0 + 2\eta(\dot{\varepsilon}_R - \dot{\varepsilon}_\theta) \tag{13}$$

Taking into account hypothesis of incompressibility of shell material $\dot{\varepsilon}_R + \dot{\varepsilon}_\theta = 0$, then similarly as in case of spherical shell instead of formulas (3)-(5), (7)-(12) we received

$$\partial v / \partial R + v / R = 0 \tag{14}$$

$$v = B(t)/R, \qquad B(t) \le 0 \tag{15}$$

$$\rho_0 \dot{v} = \frac{\partial \sigma_R}{\partial R} + \frac{\sigma_R - \sigma_\theta}{R} \tag{16}$$

$$\dot{B}(t) + \bar{\alpha} B(t) = \bar{\beta}\Sigma(t) + \bar{\gamma} \ , \tag{17}$$

$$\bar{\alpha} = \frac{2\eta(R_1^2 - R_0^2)}{\rho_0 R_1^2 R_0^2 \ln(R_1/R_0)}, \qquad \bar{\beta} = \frac{1}{\rho_0 \ln(R_1/R_0)}, \qquad \bar{\gamma} = \frac{Y_0}{\sqrt{3}\rho_0}$$

$$B(t) = \bar{\beta} e^{-\bar{\alpha} t} \int_0^t e^{\bar{\alpha} t} \Sigma(t) dt + \frac{\bar{\gamma}}{\bar{\alpha}}(1 - e^{-\bar{\alpha} t}) \tag{18}$$

$$\Sigma(0) < -\frac{\bar{\gamma}}{\bar{\beta}} = -\frac{Y_0}{\sqrt{3}_0} \ln \frac{R_1}{R_0} \tag{19}$$

$$\Sigma(t) = \Sigma_0 = const < -\frac{Y_0}{\sqrt{3}} \ln \frac{R_1}{R_0}: \quad B(t) = \frac{1}{\bar{\alpha}}(\bar{\beta}\Sigma_0 + \bar{\gamma})(1 - e^{-\bar{\alpha} t}) \tag{20}$$

$$r(R,t) = R\left\lfloor 1 + \frac{\bar{\beta}\Sigma_0 + \bar{\gamma}}{\bar{\alpha} R^2}\left(t + \frac{e^{-\bar{\alpha} t} - 1}{\bar{\alpha}} \right) \right\rfloor \tag{21}$$

$$e^{-\bar{\alpha} t_c^*} + \bar{\alpha} t_c^* = 1 - \frac{R_0^2 \bar{\alpha}^2}{\bar{\beta}\Sigma_0 + \bar{\gamma}} \tag{22}$$

4 ABOUT EXTENSION OF SPHERICAL AND CYLINDRICAL SHELLS

In case of extension shells instead of boundary conditions (6) we state next boundary conditions:

$$\sigma_R \mid_{R=R_0} = 0, \qquad \sigma_R \mid_{R=R_1} = \Sigma(t) > 0 \tag{23}$$

Easilly note that obtained solutions of one-dimensional problems about compression of thick-walled spherical and cylindrical shells from incompessible viscoplastic material may be use for extension of shells. For that sufficiently in all formulas to substitute Y_0 on $(-Y_0)$.

In case of extension of shells $C(t) \ge 0, \ B(t) \ge 0.$ Give off meaning concept of moment of collapse of shells $t_s^*, \ t_c^*$. And conditions on value of external loading in initial moment for spherical and cylindrical shells (9), (19) take respectively next form:

$$\Sigma(0) > -\frac{\gamma}{\beta} = 2Y_0 \ln \frac{R_1}{R_0}, \qquad \Sigma(0) > -\frac{\bar{\gamma}}{\bar{\beta}} = \frac{Y_0}{\sqrt{3}_0} \ln \frac{R_1}{R_0} \tag{24}$$

Consider example of extension spherical shell then loading have graduated form:

$$\Sigma(t) = \Sigma_0 H(T - t), \qquad \Sigma_0 = const > 2Y_0 \ln \frac{R_1}{R_0} \tag{25}$$

where $T -$ time of loading.

Permutate (25) in (8) and substitute Y_0 on $(-Y_0)$ we obtain

$$C(t)=\frac{(\Sigma_0-2Y_0\ln(R_1/R_0))R_0^3}{4\eta(1-(R_0/R_1)^3)}(1-e^{-\alpha t}),\qquad 0\le t\le T \tag{26}$$

$$C(t)=C(T)+\frac{\gamma}{\alpha}(e^{-\alpha T}-e^{-\alpha t}),\qquad T<t\le t_s^s \tag{27}$$

In formule (27) $C(T)$ determine from (26) and t_s^s (moment of stopping of extension for spherical shell) determine from (27) provided that $C(t_s^s)=0$:

$$t_s^s=-\frac{1}{\alpha}\ln(\frac{\alpha}{\gamma}C(T)+e^{-\alpha T}) \tag{28}$$

Function which determine formulas (26) and (27) schematically show on figure 2.

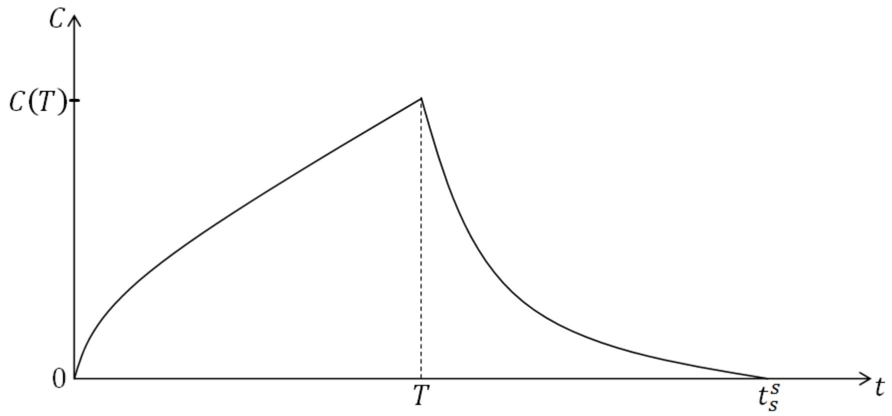

Figure 2

Distribution of ring deformation in moment of stopping of extension for spherical shell $t=t_s^s$ describe by next formule:

$$\varepsilon_\theta\big|_{t=t_s^s}=\frac{1}{R^3}\left\lfloor (t_s^s(1-e^{-\alpha T})+Te^{-\alpha T})\frac{\Sigma_0-2Y_0\ln(R_1/R_0)}{4\eta(1-(R_0/R_1)^3)}R_0^3+\frac{\gamma}{\alpha}(t_s^s-T)e^{-\alpha T}\right\rfloor$$

In case of extension of cylindrical shell under action of loading graduated form

$$\Sigma(t)=\Sigma_0 H(T-t),\qquad \Sigma_0=const>\frac{Y_0}{\sqrt{3}}\ln\frac{R_1}{R_0} \tag{29}$$

we find, substitude Y_0 on $(-Y_0)$, that

$$B(t)=\frac{(\Sigma_0-Y_0\ln(R_1/R_0)/\sqrt{3})R_0^2}{2\eta(1-(R_0/R_1)^2)}(1-e^{-\bar{\alpha}t}),\qquad 0\le t\le T \tag{30}$$

$$B(t) = B(T) + \frac{\bar{\gamma}}{\bar{\alpha}}(e^{-\bar{\alpha}T} - e^{-\bar{\alpha}t}), \qquad T < t \le t_c^s \tag{31}$$

In formule (31) $B(T)$ determine from (30) and t_c^s (moment of stopping of extension for spherical shell) determine from (31) provided that $B(t_c^s) = 0$:

$$t_c^s = -\frac{1}{\bar{\alpha}} \ln(\frac{\bar{\alpha}}{\bar{\gamma}} B(T) + e^{-\bar{\alpha}T}) \tag{32}$$

Distribution of ring deformation in moment of stopping of extension for cylindrical shell $t = t_c^s$ describe by next formule:

$$\varepsilon_\theta |_{t=t_c^s} = \frac{1}{R^2}\left[(t_c^s(1-e^{-\bar{\alpha}T}) + Te^{-\bar{\alpha}T}) \frac{\Sigma_0 - Y_0 \ln(R_1/R_0)/\sqrt{3}}{2\eta(1-(R_0/R_1)^2)} R_0^2 + \frac{\bar{\gamma}}{\bar{\alpha}}(t_c^s - T)e^{-\bar{\alpha}T} \right]$$

5 GENERALIZATION OF ZABABAXIN PROBLEM

Consider spherical layer of viscous incompressible liquid. Internal and external radius change in time on low $r_0 = r(R_0, t)$ and $r_1 = r(R_1, t)$ respectively, where R_0 and R_1 - internal and external radius of layer in initial time $t = 0$ (figure 1).

Constitutive equations of model Navier-Stokes in one-dimensional spherical case are:

$$\sigma_R = -p + 2\eta\dot{\varepsilon}_R, \qquad \sigma_\theta = -p + 2\eta\dot{\varepsilon}_\theta \tag{33}$$

Here $p = -(\sigma_R + 2\sigma_\theta)/3 -$ pressure in liquid; the other notations coincide with introduced above for viscoplastic media.

Put next boundary conditions:

$$\sigma_R |_{R=R_1} = -P_1(t) < 0, \qquad \sigma_R |_{R=R_0} = -P_0(t) < 0 \tag{34}$$

Here $P_1(t), P_0(t) -$ pressure on external and internal surfaces of spherical layer.

Substitute (33), (4) to equation of momentum (5), and integrate over R with taking account of boundary conditions (34), we get ordinary differential equation for function $C(t)$:

$$\dot{C}(t) + \alpha C(t) = \beta(P_0(t) - P_1(t)) \tag{35}$$

where

$$\alpha = \frac{4\eta(1/R_0^3 - 1/R_1^3)}{\rho_0(1/R_0 - 1/R_1)}, \qquad \beta = \frac{1}{\rho_0(1/R_0 - 1/R_1)}$$

Solution of equation (35) with initial condition $v|_{t=0} = 0$ (i.e. $C(0) = 0$) is

$$C(t) = \beta e^{-\alpha t} \int_0^t e^{\alpha t} \left(P_0(t) - P_1(t)\right) dt, \qquad v(R,t) = \frac{C(t)}{R^2}$$

Under conditions $R_1 \to +\infty$, $P_0(t) = 0$, $P_1(t) = P_\infty = const$ (Zababaxin problem [6]) we have:

$$v(R,t) = -P_\infty \frac{R_0^3}{4\eta R^2}\left(1 - \exp(-\frac{4\eta}{\rho_0 R_0^2} t)\right)$$

Solution for cylindrical layer is

$$B(t) = \bar{\beta} e^{-\bar{\alpha} t} \int_0^t e^{\bar{\alpha} t} \left(P_0(t) - P_1(t)\right) dt, \qquad v(R,t) = \frac{B(t)}{R}$$

where

$$\bar{\alpha} = \frac{2\eta(1/R_0^2 - 1/R_1^2)}{\rho_0 \ln(R_1 / R_0)}, \qquad \bar{\beta} = \frac{1}{\rho_0 \ln(R_1 / R_0)}$$

6 CONCLUSIONS

Obtained analytical solutions may be used, in particular, for testing computational programs and estimate of effectiveness of new numerical methods, as it is ware made in paper [8] used results of paper [1].

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